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A UNIFIED FRAMEWORK FOR APPROXIMATION IN INVERSE
PROBLEMS FOR DISTRIBUTED PARAMETER SYSTEMS

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# A UNIFIED FRAMEWORK FOR APPROXIMATION IN INVERSE

# PROBLEMS FOR DISTRIBUTED PARAMETER SYSTEMS*

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## ABSTRACT

We present a theoretical framework that can be used to treat approximation techniques for very general classes of parameter estimation problems involving distributed systems that are either first or second order in time. Using the approach developed, one can obtain both convergence and stability (continuous dependence of parameter estimates with respect to the observations) under very weak regularity and compactness assumptions on the set of admissible parameters. This unified theory can be used for many problems found in the recent literature and in many cases offers significant improvements to existing results.
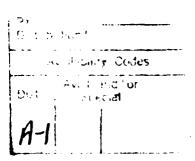
## 1. Introduction

An important class of scientific problems related to the control of flexible space structures entails the estimation of parameters in distributed or partial differential equation models. These inverse or parameter identification problems arise in several contexts. A rather obvious class of such problems involves the development and identification of physical models for flexible structures — e.g., see [BC1], [BC2], [BCR], [BR2], [BPR], [BWIC], [BFW]. In these problems one typically investigates models from structural mechanics which contain parameters representing elastic properties (such as stiffness, damping) of materials. The inverse problems then consist of estimating these parameters using data obtained from observations of the system response to dynamic loading or displacement. The partial differential equation models usually are second order in time, higher order in space (for example, the Euler-Bernoulli or Timoshenko beam models and their higher dimensional analogues for plates, etc.). The related inverse problems can prove quite challenging, but are extremely important as a precursor to and integral part of the design of control strategies for space structures.

A second, less obvious, class of inverse problems that are important in the study of space structures involves those arising in material testing, in particular, in the area of nondestructive evaluation of materials. These problems, such as those related to the detection of structural flaws (broken fibers, delaminations, cracks) in fiber reinforced composite materials, have been addressed at NASA (for example, in connection with evaluation of the solid rocket motors for the

Space Shuttle) using a variety of techniques (acoustic, thermal, etc.). One technique [HWW], [BKO] entails the use of thermal diffusivity properties (a thermal tomography) to characterize materials. The resulting inverse problems involve using boundary observations for the estimation of thermal diffusion coefficients and interior boundaries of the domain (i.e. the geometrical structure of the system domain) for the heat equation in two or three space dimensions. These so-called domain identification problems thus constitute an important class of parabolic system identification problems related to space structures.

In this paper we present a general convergence/stability (continuous dependence on observations) framework for approximation methods to treat parameter identification problems (see [BCK]) involving distributed parameter systems. This new parameter estimation convergence framework combines a weak version of the system (in terms of sesquilinear forms in the spirit used in [BK1]) with the resolvent convergence form of the Trotter-Kato approximation theorem [P], [BK2]. The very general convergence results depend (in addition, of course, to the usual properties for the associated approximating subspaces) on three properties of the parameter ($q \in Q$) dependent sesquilinear form $\sigma(q)(\cdot, \cdot)$ describing the system: (A) <u>continuity</u> (with respect to the parameter); (B) uniform (in the parameter) <u>coercivity</u>; and (C) uniform (in the parameter) <u>boundedness</u>. The approach permits one to give convergence and stability arguments in inverse problems under extremely weak compactness assumptions on the admissible parameter spaces Q (equivalent to those in typical variational or weak approaches--see [B1], [BCR]) without requiring knowledge of smoothness of solutions usually a part

of the variational and general finite element type arguments. Thus this approach combines in a single framework the best features of a semigroup approximation approach (i.e., the Trotter-Kato theorem) with the best features of a variational approach (weak assumptions on Q). The weakening of the compactness criteria on Q is of great computational importance since the constraints associated with these criteria should be implemented in problems (see [BI] for further discussion and examples).

An additional significant feature of the approach described in this paper is that for first order systems it yields directly a stronger convergence (in the spatial coordinate) than one readily obtains without extra effort in either the convergence arguments using the usual operator convergence form of the Trotter-Kato formulation (e.g., see Theorem 2.2 of [BCK] and Theorem 2.4 of [KW]) or the finite element type variational arguments (e.g., see, Theorem 4.1 of [BRI] and the related remarks in [BKL1]).

While the approach we present here does involve coercivity of certain sesquilinear forms associated with the system dynamics, its applicability is not restricted to parabolic systems which generate analytic semigroups. As we shall demonstrate in Section 3 below, it can be used to treat problems in which the underlying semigroup is not analytic (e.g., those involving Euler-Bernoulli equations for beams with various types of damping-viscous, Kelvin-Voigt, spatial hysteresis), improving substantially on some of the currently known results for these problems. With appropriate modifications, this theoretical framework can be generalized to also allow an elegant treatment of problems involving functional partial differential equations, e.g., beams with Boltzmann damping (i.e., time hysteresis) -- see [BFW].

## 2. First Order Systems

We consider first order systems dependent on parameters $q \in Q$ described by an abstract equation

$$\dot{u}(t) = A(q)u(t) + F(t,q)$$
$$u(0) = u_0(q)$$
(2.1)

in a Hilbert space H. The admissible parameter space Q is a metric space with metric d and for $q \in Q$, we assume that $A(q)$ is the infinitesimal generator of a $C_0$ semigroup $T(t;q)$ on H. We assume that we are given observations $\bar{u}_i \in H$ for the mild solution values $u(t_i,q)$ of (2.1); i.e., we solve (2.1) in the sense

$$u(t;q) = T(t;q)u_0(q) + \int_0^t T(t-s;q)F(s,q)ds$$
(2.2)

in H. *We then consider the least squares identification (ID) problem of minimizing over* $q \in Q$ *the functional*

$$J(q) = \sum_i |u(t_i;q) - \bar{u}_i|^2.$$
(2.3)

Such problems are, in general, infinite dimensional in both the state u and the parameter q and thus one must consider a sequence of computationally tractable approximating problems. These can be, for our purposes, best described in terms of parameter dependent sesquilinear forms $\sigma(q)(\cdot,\cdot)$ associated with (2.1) or (2.2) (i.e., forms which define the operators $A(q)$ in (2.1)). For more details and examples in the parabolic case, we refer the reader to [BK1]. Briefly, let V and H be Hilbert spaces with V continuously

and densely imbedded in H. Denote a family of parameter dependent sesquilinear forms by $\sigma(q)$: $V \times V \to \mathbb{C}$, $q \in Q$. We assume that $\sigma$ possesses the following properties:

(A) <u>Continuity</u>: For $q, \bar{q} \in Q$, we have for all $\phi, \psi \in V$

$$|\sigma(q)(\phi,\psi) - \sigma(\bar{q})(\phi,\psi)| \leq d(q,\bar{q})|\phi|_V|\psi|_V.$$

(B) <u>Coercivity</u>: There exist $c_1 > 0$ and some real $\lambda_0$ such that for $q \in Q$, $\phi \in V$ we have

$$\sigma(q)(\phi,\phi) + \lambda_0|\phi|_H^2 \geq c_1|\phi|_V^2.$$

(C) <u>Boundedness</u>: There exist $c_2 > 0$ such that for $q \in Q$, $\phi, \psi \in V$ we have

$$|\sigma(q)(\phi,\psi)| \leq c_2|\phi|_V|\psi|_V.$$

Under these assumptions, $\sigma$ defines in the usual manner (e.g., see [K], [S]) operators $A(q)$ such that $\sigma(q)(\phi,\psi) = \langle -A(q)\phi,\psi\rangle_H$ for $\phi \in \text{dom}(A(q))$, $\psi \in V$ with $\text{dom}(A(q))$ dense in V. Furthermore, $A(q)$ is the generator of an analytic semigroup $T(t;q)$ on H (indeed, $A(q)$ is sectorial with $(\lambda I - A(q))\text{dom}(A(q)) = H$) and mild solutions of (2.1) possess additional regularity (e.g. see Chap. 4 of [P] and Chap. 4 of [5]) under appropriate assumptions on F. Property (B) guarantees that for $\lambda \geq \lambda_0$ the resolvent operator $R_\lambda(A(q)) \equiv (\lambda I - A(q))^{-1}$ exists as a bounded operator on H; in fact, it follows readily from (B) that (assuming $|\cdot|_H \leq k|\cdot|_V$) we have

$$c_1|R_\lambda(A(q))\psi|_V^2 \leq \langle\psi, R_\lambda(A(q))\psi\rangle_H$$
$$\leq |\psi|_H|R_\lambda(A(q))\psi|_H$$
$$\leq k^2|\psi|_V|R_\lambda(A(q))\psi|_V$$

and hence for $\psi \in V$

$$|R_\lambda(A(q))\psi|_V \leq \frac{k^2}{c_1}|\psi|_V \tag{2.4}$$

while for $\psi \in H$ we find

$$|R_\lambda(A(q))\psi|_H \leq \frac{k^2}{c_1}|\psi|_H. \tag{2.5}$$

In addition, one can use (B) and (A) to argue that $q \to R_\lambda(A(q))$ is a continuous mapping from Q to V. It is these ideas that can be modified to give resolvent convergence in the approximation schemes which we introduce next.

We consider Galerkin type approximations in the context of sesquilinear forms (e.g., see [BK1] for further details). Let $H^N$ be a family of finite dimensional subspaces of H satisfying $P^N z \to z$ for $z \in H$ where $P^N$ is the orthogonal projection of H onto $H^N$. We further assume that $H^N \subset V$ and that the family possesses certain V-approximation properties to be specified below. If we now consider the restriction of $\sigma(q)(\cdot,\cdot)$ to $H^N \times H^N$, we obtain operators $A^N(q): H^N \to H^N$ which, because of (B), satisfy a uniform dissipative inequality and can be shown to generate semigroups $T^N(t;q)$ in $H^N$. These are then used to define approximating systems for (2.2):

$$u^N(t;q) = T^N(t;q)P^N u_0(q) + \int_0^t T^N(t-s;q)P^N F(s,q)ds. \tag{2.6}$$

One thus obtains a sequence of approximating ID problems consisting of minimizing over Q

$$J^N(q) = \sum_i |u^N(t_i;q) - \bar{u}_i|^2. \tag{2.7}$$

In problems where Q is infinite dimensional (the usual case in many inverse problems of interest), on must also make approximations $Q^M$ for Q (see [BD] [BR2] for details). To include this aspect of the problem in our discussion here does not entail any essential mathematical difficulties. Since it would increase the notational clutter and provide no new mathematical insight, we do not pursue the theory in that generality.

To obtain convergence and continuous dependence (of parameter estimates with respect to observations) results for the solutions $\bar{q}^N$ of minimizing $J^N$ in (2.7), it suffices under the assumption that (Q,d) is a compact space (see [B1]) to argue: for <u>arbitrary</u> $\{q^N\} \subset Q$ with $q^N \to q$ we have $u^N(t;q^N) \to u(t;q)$ for each t. Under reasonable assumptions on F and $u_0$, this can be argued if one first shows that $T^N(t;q^N)P^N z \to T(t;q)z$ for arbitrary $q^N \to q$ and $z \in H$. To do this one can use several versions of the Trotter- Kato theorem [P], [BK2]. We state precisely the "resolvent convergence form" of this theorem in a form that is general enough for us to use in several contexts subsequently in this paper.

Let X and $X^N$, N = 1,2, ..., be Hilbert spaces, $X^N \subset X$, and let $P^N: X \to X^N$ be the orthogonal projection of X onto $X^N$. We assume the $X^N$'s approximate X in the sense that $P^N x \to x$ for all $x \in X$. Then we have:

**Theorem 2.1:** *Let $A^N$ and A be infinitesimal generators of $C_0$ semigroups $S^N(t)$ and S(t) on $X^N$ and X, respectively, satisfying:*

   (i)   *There exist constants $\omega$, M such that $|S^N(t)| \leq Me^{\omega t}$ for each N;*

   (ii)  *There exists $\lambda \in \rho(A) \cap_{N=1}^{\infty} \rho(A^N)$ with Re $\lambda > \omega$ such that*

            $R_\lambda(A^N)P^N x \to R_\lambda(A)x$ *for each $x \in X$.*

*Then*

(iii) *For each* $x \in X$, $S^N(t)P^Nx \to S(t)x$ *uniformly in t on any compact interval* $[0, t_1]$.

We may assume without loss of generality that the constants $\omega$ and M of (i) are chosen so that $S(t)$ also satisfies the bound in (i).

There is an alternate version (we shall for obvious reasons refer to this version as the "operator convergence form") of the Trotter-Kato theorem which has been frequently used [BCK], [BC1], [BC2], [R] in parameter estimation problems. This version replaces condition (ii) above by the condition:

(ii') *There exists a set $\mathcal{D}$ dense in X such that for some $\lambda$, $(\lambda I - A)\mathcal{D}$ is dense in X and $A^NP^Nx \to Ax$ for all $x \in \mathcal{D}$.*

As we have already noted, it is the resolvent convergence form of this theorem which we shall wish to make use of in our theoretical framework since, as we shall show, for our first order systems condition (ii) will follow readily from (A), (B), (C) and a condition on how $H^N$ approximates H. The entire theory will require only that Q be compact in the metric d used in the continuity statement (A) for the sesquilinear form. This will, in general, be much weaker than that compactness required for use in proofs employing condition (ii'), since the requirements on "$q^N \to q$" to insure $A^N(q^N)P^Nz \to A(q)z$ typically involve convergence of some derivatives of the $q^N$. For example, in first order parabolic equations containing variable coefficients to be estimated, this results in a requirement that Q be compact in $H^1$ whereas use of the sesquilinear form approach developed in this paper requires only compactness of Q in C or $L^\infty$ (see [B1] for further discussion of this point).

Before turning to our convergence arguments, we state the convergence properties required of the approximating subspaces $H^N$ alluded to above. Throughout our discussions, we shall assume:

(C1)   For each $z \in V$, there exists $\hat{z}^N \in H^N$ such that $|z - \hat{z}^N|_V \to 0$ as $N \to \infty$.

**Theorem 2.2:** *Let conditions* (A), (B), (C) *and* (C1) *hold and* $q^N \to q$ *in* Q. *Then for* $\lambda = \lambda_0$, $R_\lambda(A^N(q^N))P^N z \to R_\lambda(A(q))z$ *in the* V *norm for any* $z \in H$.

**Proof:** First note that since $A^N(q^N)$ results from the restriction of $\sigma$ to $H^N \times H^N$, we can choose $\lambda = \lambda_0$ in (B) so that $\lambda \in \rho(A(q)) \cap_{N=1}^{\infty} \rho(A^N(q^N))$ with $\lambda > \omega$ (and indeed bounds similar to those of (2.4), (2.5) hold for $R_\lambda(A^N(q^N))P^N$).

Let $z \in H$ be arbitrary and for notational convenience put $w = w(q) \equiv R_\lambda(A(q))z$ and $w^N = w^N(q^N) \equiv R_\lambda(A^N(q^N))P^N z$. Since $w \in \text{dom}(A(q)) \subset V$, we may use condition (C1) to define a sequence $\{\hat{w}^N\}$ satisfying $|\hat{w}^N - w|_V \to 0$ as $N \to \infty$.

We wish to show $|w^N - w|_V \to 0$. If we argue that $|w^N - \hat{w}^N|_V \to 0$, the desired results follow immediately from the triangle inequality.

Let $z^N \equiv w^N - \hat{w}^N$ so that $z^N \in H^N \subset V$. Using $\langle \cdot, \cdot \rangle$ to denote the inner product in H, we find

$$\sigma(q)(w, z^N) = \langle -A(q)w, z^N \rangle = \langle (\lambda - A(q))w, z^N \rangle - \lambda \langle w, z^N \rangle$$
$$= \langle z, z^N \rangle - \lambda \langle w, z^N \rangle$$

and

$$\sigma(q^N)(w^N,z^N) = \langle -A^N(q^N)w^N,z^N \rangle = \langle (\lambda - A^N(q^N))w^N,z^N \rangle - \lambda \langle w^N,z^N \rangle$$

$$= \langle P^N z,z^N \rangle - \lambda \langle w^N,z^N \rangle$$

$$= \langle z,z^N \rangle - \lambda \langle w^N,z^N \rangle.$$

Hence we have

$$\sigma(q)(w,z^N) = \sigma(q^N)(w^N,z^N) + \lambda \langle w^N - w,z^N \rangle.$$

Using this with (B) we obtain

$$c_1|z^N|_V^2 \leq \sigma(q^N)(z^N,z^N) + \lambda|z^N|_H^2 = \sigma(q^N)(w^N,z^N) - \sigma(q^N)(\hat{w}^N,z^N) + \lambda|z^N|_H^2$$

$$= \sigma(q)(w,z^N) + \lambda \langle w-w^N,z^N \rangle - \sigma(q^N)(\hat{w}^N,z^N) + \lambda|z^N|_H^2$$

$$= \sigma(q)(w,z^N) - \sigma(q^N)(w,z^N) + \sigma(q^N)(w-\hat{w}^N,z^N)$$

$$+ \lambda(\langle w-w^N,z^N \rangle + |z^N|_H^2).$$

If we use (A) and (C), we thus have

$$c_1|z^N|_V^2 \leq d(q,q^N)|w|_V|z^N|_V + c_2|w-\hat{w}^N|_V|z^N|_V + \lambda(\langle w-w^N,z^N \rangle + |z^N|_H^2).$$

Finally, since

$$\langle w-w^N,z^N \rangle = \langle w-\hat{w}^N,z^N \rangle + \langle \hat{w}^N-w^N,z^N \rangle = \langle w-\hat{w}^N,z^N \rangle - \langle z^N,z^N \rangle,$$

the above inequality may be written

$$c_1|z^N|_V^2 \leq d(q,q^N)|w|_V|z^N|_V + c_2|w-\hat{w}^N|_V|z^N|_V + \lambda \langle w-\hat{w}^N,z^N \rangle$$

$$\leq d(q,q^N)|w|_V|z^N|_V + c_2|w-\hat{w}^N|_V|z^N|_V + |\lambda||w-\hat{w}^N|_H|z^N|_H.$$

Thus, using $|\cdot|_H \leq k|\cdot|_V$ we find

$$c_1|z^N|_V \leq d(q,q^N)|w|_V + (c_2 + |\lambda|k^2)|w-\hat{w}^N|_V,$$

which, from the definition of $\{\hat{w}^N\}$ and the fact that $q^N \to q$, yields the desired convergence $z^N = w^N - \hat{w}^N \to 0$ in V norm.

Since V norm convergence is stronger than H-norm convergence, we can use the results of this theorem along with Theorem 2.1 for $X = H$, $X^N = H^N$, $P^N = P^N$, $A^N = A^N(q^N)$, $A = A(q)$ to obtain immediately that $T^N(t;q^N)P^N z \to T(t;q)z$ for $z \in H$, the convergence being in the H norm. This along with (2.2) and (2.6) can be used to argue the desired convergence $u^N(t;q^N) \to u(t;q)$ in the H norm (assuming, of course, appropriate smoothness of F and $u_0$ in $q \in Q$). However, with no additional assumptions and only a little extra effort, one can obtain this convergence in the V norm. As we shall explain below, this stronger convergence is often immensely useful in parameter estimation problems where the observations may be continuous in the V norm, but not in the H norm.

**Theorem 2.3:** *Under the hypotheses of Theorem 2.2, we have for each $z \in H$, $T^N(t;q^N)P^N z \to T(t;q)z$ in the V norm for $t > 0$, uniformly in t on compact subintervals.*

**Proof:** To establish these results, we first apply the Trotter-Kato theorem in the space $X = V$ with $X^N = H^N$. Let $P_V^N \colon V \to H^N$ denote the orthogonal projection of V onto $H^N$ in the V inner product. Then hypothesis (C1) implies $P_V^N f \to f$ in the V norm for any $f \in V$ as well as $P^N f \to f$ in the H norm for any $f \in H$.

To carry out our arguments, we shall employ several bounds for resolvents similar to but sharper than those of (2.4), (2.5). These bounds follow from

results in Tanabe (c.f. Lemma 6.1 of Chapter 3 in [T]); we note that our operator A can be extended to A: $V \to V^*$ as in [T] so that the results given there are applicable as used below. (In [T], the condition (B) is assumed to hold with $\lambda_0 = 0$ and we, without loss of generality, state our bounds for this case.) The first bound we shall use yields for $f \in V$

$$|R_\lambda(A^N(q^N))f|_V \leq \frac{c}{|\lambda|^{1/2}} |f|_H \tag{2.8}$$

where c is independent of N. Using this with $f = P_V^N z - P^N z$ for $z \in V$ along with the results of Theorem 2.2 we obtain $R_\lambda(A^N(q^N))P_V^N z \to R_\lambda(A(q))z$ in V norm for any $z \in V$. This is (ii) of Theorem 2.1 with $X = V$, $X^N = H^N$ and $P^N = P_V^N$.

To obtain the uniform stability bound (i) of Theorem 2.1 in the V-norm, we make use of another bound that follows from [T]. From the last bound of Lemma 6.1, Chap. 3 of [T] one may readily argue that for $f \in V$

$$|R_\lambda(A^N(q^N))f|_V \leq \frac{M_1}{|\lambda|}|f|_V \tag{2.9}$$

where $M_1$ is independent of N. The arguments behind (2.9) involve using (3.51) of [T] for $A^*$ and observing that for $\gamma \in V$, $v \in V^*$ arbitrary so that $R_\lambda(A)\gamma \in V = V^{**}$ we have (using the usual notation for the duality product)

$$(R_\lambda(A)\gamma, v)_{V^{**}, V^*} = (v, R_\lambda(A)\gamma)_{V^*, V} = (R_\lambda(A^*)v, \gamma)_{V^*, V}$$
$$\leq |R_\lambda(A^*)v|_{V^*}|\gamma|_V \leq \frac{M_1}{|\lambda|} |\gamma|_V |v|_{V^*}.$$

Then $|R_\lambda(A)\gamma|_{V^{**}} \leq \frac{M_1}{|\lambda|} |\gamma|_V$ and hence, identifying V and $V^{**}$, we obtain

$|R_\lambda(A)\gamma|_V \leq \frac{M_1}{|\lambda|} |\gamma|_V$ for $\gamma \in V$.

We may use the bound (2.9) in the identity

$$T^N(t;q^N) = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} R_\lambda(A^N(q^N)) d\lambda, \tag{2.10}$$

where $\Gamma$ is a contour (about $\lambda_0$) similar to that given in [P, p. 63], to argue for $z \in H^N$

$$|T^N(t;q^N)z|_V \leq M c^{\lambda_0 t} |z|_V \tag{2.11}$$

where $M$ is independent of $N$. Thus one obtains $|T^N(t;q^N)|_V \leq M c^{\lambda_0 t}$. (These results are essentially given in [T]; one just needs to check the arguments to ascertain that the constant $M$ depends on $A^N$ only through the bounds $c_1$ and $c_2$ of conditions (B) and (C) and the sector angle $\theta$ for $A^N$ -- where $\theta$ again depends only on $c_1$ and $c_2$.)

We may then apply Theorem 2.1 as indicated to obtain $T^N(t;q^N)P_V^N z \rightarrow T(t;q)z$ in $V$ norm for every $z \in V$. It remains to argue that $T^N(t;q^N)P^N z \rightarrow T(t;q)z$ in $V$ norm for every $z \in H$.

If we use the bound (2.8) in (2.10), arguments similar to those used to establish (2.11) can be made to give

$$|T^N(t;q^N)z|_V \leq \tilde{M} c^{\lambda_0 t} t^{-1/2} |z|_H, \quad t > 0, \quad z \in H^N, \tag{2.12}$$

where again $\tilde{M}$ is independent of $N$.

Recalling that $V$ is dense in $H$, then given $z \in H$ and $\epsilon > 0$, we may choose $z_V \in V$ such that $|z_V - z|_H < \epsilon$. Then noting that $T(t;q)z \in V$ for $t > 0$, we have

$$|T^N(t;q^N)P^N z - T(t;q)z|_V \leq |T^N(t;q^N)(P^N z - P_V^N z_V)|_V$$

$$+ |T^N(t;q^N)P_V^N z_V - T(t;q)z_V|_V + |T(t;q)(z_V - z)|_V.$$

From (2.12) we see that the first term in the right side of this inequality is bounded by $\widehat{M}e^{\lambda_0 t} t^{-1/2} |P^N z - P_V^N z_V|_H$ which, as $N \to \infty$, approaches $\widehat{M}e^{\lambda_0 t} t^{-1/2} |z - z_V|_H$. The third term is also bounded by a similar expression, while the second term approaches zero as $N \to \infty$ by our previous arguments. We thus have established the convergence claimed in the statement of Theorem 2.3.

Among the examples that can be treated immediately with the above theory are the usual parabolic systems (see [BK1], [L]). To illustrate these ideas, consider the estimation problem for the standard Dirichlet boundary value problem for one-dimensional parabolic systems. That is,

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[q_1 \frac{\partial u}{\partial x}\right] + \frac{\partial}{\partial x}(q_2 u) + q_3 u \quad \text{on} \quad \Omega = (0,1)$$

$$u(t,0) = u(t,1) = 0,$$

with $q = (q_1, q_2, q_3)$ to be chosen (via a least squares criterion) from Q, a compact subset of $C(\Omega) \times C(\Omega) \times C(\Omega)$. The state spaces are $H = H^0(\Omega)$, $V = H_0^1(\Omega)$ and the weak form of the equation is given by

$$\langle \dot{z}(t), \psi \rangle + \sigma(q)(z(t), \psi) = 0, \quad \psi \in V$$

where the sesquilinear form is defined as

$$\sigma(q)(\varphi, \psi) = \langle q_1 D\varphi, D\psi \rangle + \langle q_2 \varphi, D\psi \rangle - \langle q_3 \varphi, \psi \rangle,$$

with $D = \partial/\partial x$. One can readily verify that conditions (A) and (C) hold, while the coercivity condition (B) is valid if we assume elements of Q satisfy $q_1(x) \geqslant \nu > 0$ for some constant $\nu$.

In this case one can specify the domain of $A(q)$ by

$$\text{dom } A(q) = \{\varphi \in H_0^1(\Omega) \mid q_1 D\varphi + q_2\varphi \in H^1(\Omega)\}$$

and $A(q)\varphi = D(q_1 D\varphi) + D(q_2\varphi) + q_3\varphi$. For approximating elements one can use either piecewise cubic or piecewise linear B-splines modified to satisfy the Dirichlet boundary conditions.

We note that the theory above only requires compactness of Q in the $[C(\Omega)]^3$ topology along with relatively weak smoothness assumptions when compared with other approaches (see [BK3], [BKL1], [BKL2], [BCK]). Furthermore we obtain $V = H_0^1(\Omega)$ convergence of the states. Hence the theory is complete for least squares criterion involving pointwise (in x) observations of the state. This is obtained for little extra effort when compared with the efforts usually required to obtain the stronger convergence (see [BCK], [BKL1]).

Other parabolic examples of great practical importance can be readily treated in the context of this framework. For example, the 2-D thermal tomography problem mentioned in the Introduction and discussed in [BKO] results in a boundary identification problem for parabolic systems. With a standard transformation these problems can be readily treated theoretically and computationally with the approach of this paper. Details can be found in [BKO].

Another class of problems of interest involves estimation of coefficients

in the Fokker-Planck or forward Kolmogorov equations (see [B2]) for the case where the Markov transition process for growth is time invariant in size/age structured population models. In this case the equations have the form

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(q_1 u) = \frac{\partial^2}{\partial x^2}(q_2 u) - q_3 u, \quad x_0 < x < x_1,$$

with boundary conditions

$$\left[ q_1 u - \frac{\partial}{\partial x}(q_2 u) \right]^{x=x_0} = \int_{x_0}^{x_1} q_4(\xi) u(t,\xi) d\xi$$

$$\left[ q_1 u - \frac{\partial}{\partial x}(q_2 u) \right]^{x=x_1} = 0.$$

The associated sesquilinear form is given by (here $H = H^0(x_0, x_1)$, $V = H^1(x_0, x_1)$)

$$\sigma(q)(\varphi, \psi) = -\langle q_1 \varphi - D(q_2 \varphi), D\psi \rangle - \psi(x_0) R(q_4)(\varphi) + \langle q_3 \varphi, \psi \rangle$$

with $R(q_4)(\varphi) \equiv \int_{x_0}^{x_1} q_4(\xi)\varphi(\xi) d\xi$. Under appropriate assumptions on $Q \subset L_\infty(\Omega) \times W_\infty^{(1)}(\Omega) \times L_\infty(\Omega) \times L_\infty(\Omega)$, $\Omega = (x_0, x_1)$, (see [B2] for details) one can readily argue that hypotheses (A), (B), (C) hold. Thus these problems also fall within the purview of the theory developed here.

## 3. Second Order Systems

The ideas developed in the previous section can be applied to parameter estimation problems involving second order systems of the form

$$\ddot{u}(t) + B(q)\dot{u}(t) + A(q)u(t) = f(t) \tag{3.1}$$

in a Hilbert space H where the operators $A(q)$ and $B(q)$ are defined via parameter dependent sesquilinear forms in a manner similar to that of Section 2. We use the general approach given in [S]; we again assume we are given a Hilbert space $V \subset H$ that is continuously and densely imbedded in H. Let $\sigma_1(q): V \times V \rightarrow \mathbb{C}$ be a symmetric sesquilinear form satisfying conditions (A) and (C) of Section 2 and the V coercivity condition (B) with $\lambda_0 = 0$. Then for each $q \in Q$ we may define continuous linear operators $A(q): V \rightarrow V^*$ given by

$$\sigma_1(q)(\phi,\psi) = (A(q)\phi,\psi)_{V^*,V} \tag{3.2}$$

for $\phi,\psi \in V$, where as customary, we have identified the pivot space H with its dual $H^*$, i.e., $V \subset H = H^* \subset V^*$ and the duality product $(\cdot,\cdot)_{V^*,V}$ is the unique extension by continuity of the scalar product $\langle\cdot,\cdot\rangle$ of H from $H \times V$ to $V^* \times V$ (e.g., see [S] for details). Of course, as in Section 2, we may also view $A(q)$ as a densely defined operator in H where $\sigma_1(q)(\phi,\psi) = \langle A(q)\phi,\psi\rangle_H$ for $\phi \in \text{dom } A(q)$, $\psi \in V$. The sesquilinear form $\sigma_1$ and its associated operator A will correspond to the "stiffness" operator in the examples we treat below.

A second sesquilinear form will give rise to the "damping" term in our examples. We assume we are given a (not necessarily symmetric) sesquilinear form $\sigma_2(q): V \times V \rightarrow \mathbb{C}$ satisfying conditions (A) and (C) and the <u>semicoercivity</u>

condition

(B') <u>H-semicoercivity</u>: There exists $b \geqslant 0$ such that for $q \in Q$ and $\psi \in V$ we have

$$\sigma_2(q)(\psi,\psi) \geqslant b|\psi|_H^2. \tag{3.3}$$

As before, under condition (C) on $\sigma_2$, we may define a continuous linear operator $B(q): V \rightarrow V^*$ given by

$$\sigma_2(q)(\phi,\psi) = (B(q)\phi,\psi)_{V^*,V}. \tag{3.4}$$

Again, we may alternatively view $B(q)$ as being densely defined in H. However, since we may wish in general to view equation (3.1) as an equation in $V^*$ (again following classical formulations [S], [L], [T]) we shall interpret $A(q)$ and $B(q)$ as members of $\mathcal{L}(V,V^*)$. We note that from conditions (B) and (C) on $\sigma_1$, the form $\sigma_1(q)$ is, for each $q \in Q$, equivalent to the norm inner product in V; indeed, we can use $\sigma_1(q)$ to define a parameter dependent equivalent inner product in V. We shall use $V_q$ to denote the Hilbert space consisting of the elements of V equipped with this inner product.

We shall rewrite equation (3.1) in first order form and to that end we define the product spaces $V = V \times V$ and $\mathcal{H} = V \times H$. If $\langle \cdot,\cdot \rangle_V$ denotes the inner product in V, we may use $\sigma_1$ and $\sigma_2$ to define a sesquilinear form $\sigma : V \times V \rightarrow \mathbb{C}$ given by

$$\sigma(q)((u,v),(\phi,\psi)) = -\langle v,\phi \rangle_V + \sigma_1(q)(u,\psi) + \sigma_2(q)(v,\psi). \tag{3.5}$$

We may then write equation (3.1) in weak or variational form as

$$\langle \dot{w}(t),\chi \rangle_{\mathcal{H}} + \sigma(q)(w(t),\chi) = \langle F(t),\chi \rangle_{\mathcal{H}} , \quad \chi \in V, \tag{3.6}$$

where $w(t) = (u(t),v(t))$, $\chi = (\phi,\psi)$, $F(t) = (f(t),0)$ and $\langle \cdot,\cdot \rangle_{\mathcal{H}}$ is the usual product space inner product. Alternatively, we may write equation (3.1) as

$$\dot{w}(t) = A(q)w(t) + F(t) \tag{3.7}$$

where $A(q)$ is the operator associated with the sesquilinear form $\sigma$ in the usual manner.

Following this approach, we define in $\mathcal{H} = V \times H$ the operator

$$A(q) = \begin{bmatrix} 0 & I \\ -A(q) & -B(q) \end{bmatrix} \tag{3.8}$$

on dom $A(q) = \{(\phi,\psi) \in \mathcal{H} \mid \psi \in V \text{ and } A(q)\phi + B(q)\psi \in H\} \subset V$. We can readily argue that dom $A(q)$ is dense in $\mathcal{H} = V \times H$ and that, under the conditions on $\sigma_1$ and $\sigma_2$, for $\lambda > 0$ the range of $\lambda I - A(q)$ is $\mathcal{H}$ (see the arguments given below). Moreover, $A(q)$ is dissipative in $\mathcal{H}_q \cong V_q \times H$ since for $(\phi,\psi) \in \text{dom } A(q)$

$$\langle A(q)(\phi,\psi),(\phi,\psi) \rangle_{\mathcal{H}_q} = \sigma_1(q)(\psi,\phi) - \sigma_1(q)(\phi,\psi) - \sigma_2(q)(\psi,\psi)$$
$$\leqslant - b|\psi|_H^2 \leqslant 0.$$

Thus by the Lumer-Phillips theorem (e.g. see Chap. 1, Thms. 4.3, 4.6 of [P]) we find that $A(q)$ generates a semigroup of contractions on $V_q \times H$. Since $V_q$ and $V$ possess equivalent (uniformly in $q \in Q$) norms, we have that $A(q)$ is the infinitesimal generator of a $C_0$-semigroup $T(t;q)$ on $\mathcal{H} = V \times H$. If the form $\sigma_2$ satisfies (3.3) with $b > 0$, then one can show that this semigroup is uniformly exponentially stable, i.e., $|T(t;q)| \leqslant Me^{-\omega t}$ for some $M \geqslant 1$ and $\omega > 0$. Furthermore, if the H semicoercivity of (3.3) is replaced by V coercivity, i.e., $b > 0$ and the H norm is replaced by the V norm in (3.3), then $T(t;q)$ is an analytic semigroup on $\mathcal{H} = V \times H$.

Since we have $\sigma(q)(\chi,\xi) = -\langle A(q)\chi,\xi \rangle_{\mathcal{H}}$ for $\chi \in \text{dom } A(q)$ and $\xi \in V = V \times V$, one might be tempted to apply the arguments and results of Section 2 to this sesquilinear form with $V$ and $\mathcal{H}$ playing the role of the spaces V and H in the theory of Section 2 (i.e., treat this system generated by $A(q)$ as a first order system in $\mathcal{H}$ and apply without modification the theory in Section 2). If one did this, then conditions (A), (B) and (C) must be satisfied by $\sigma(q)$ using the norms of $\mathcal{H}$ and $V$. Conditions (A) and (C) pose no difficulty under the assumptions of (A) and (C) on $\sigma_1$ and $\sigma_2$ given above. However to argue that $\sigma(q)$ is $V$ coercive (condition (B) of Section 2), one finds that the H semicoercivity condition (3.3) on $\sigma_2(q)$ must be strengthened to V coercivity. While some damping forms of interest (e.g. strong Kelvin-Voigt damping -- we discuss this below) do satisfy this stronger condition, several important types of damping lead to sesquilinear forms that don't. Hence we shall modify the arguments behind Theorem 2.2 in order to enable us to treat the more general cases.

We shall throughout our discussions henceforth assume that $\sigma_1$ satisfies conditions (A), (B) with $\lambda_0 = 0$, and (C), while $\sigma_2$ satisfies conditions (A), (B'), (C). Again we shall be interested in a resolvent convergence form of the Trotter-Kato theorem and hence must consider the resolvent of the operator $A(q)$ given in (3.8).

To motivate our discussions, consider for $\lambda > 0$ the equation in $\mathcal{H}$ given by $\chi = R_\lambda(A(q))\xi$ for $\chi = (\varphi,\psi)$, $\xi = (\eta,\gamma)$. This is equivalent to solving for $\chi \in \text{dom}(A(q))$ in the equation $(\lambda I - A(q))\chi = \xi$ which may be written

$$\lambda\varphi - \psi = \eta$$

$$\lambda\psi + A(q)\varphi + B(q)\psi = \gamma . \tag{3.9}$$

If we substitute the first equation $\psi = \lambda\varphi - \eta$ of this system into the second we have

$$\lambda^2\varphi + A(q)\varphi + \lambda B(q)\varphi = \gamma + \lambda\eta + B(q)\eta \tag{3.10}$$

which must be solved for $\varphi \in V$. This suggests that we define, for $\lambda > 0$, the associated sesquilinear form $\sigma_\lambda(q): V \times V \to \mathbb{C}$ given by

$$\sigma_\lambda(q)(\varphi,\zeta) = \lambda^2\langle\varphi,\zeta\rangle_H + \sigma_1(q)(\varphi,\zeta) + \lambda\sigma_2(q)(\varphi,\zeta). \tag{3.11}$$

Since $\sigma_1$ is V-coercive and $\sigma_2$ is H-semicoercive, we find for $\varphi \in V$

$$\sigma_\lambda(q)(\varphi,\varphi) = \lambda^2|\varphi|_H^2 + \sigma_1(q)(\varphi,\varphi) + \lambda\sigma_2(q)(\varphi,\varphi)$$

$$\geqslant \lambda^2|\varphi|_H^2 + c_1|\varphi|_V^2 + \lambda b|\varphi|_H^2.$$

Hence for $\varphi \in V$

$$\sigma_\lambda(q)(\varphi,\varphi) \geqslant c_1|\varphi|_V^2 + \hat{\lambda}|\varphi|_H^2 > c_1|\varphi|_V^2 \tag{3.12}$$

where $\hat{\lambda} \equiv (\lambda^2 + \lambda b) > 0$. Thus, for $\lambda > 0$, the form $\sigma_\lambda$ is V-coercive and the equation (3.10) is solvable for $\varphi \in V$ for any given $\xi = (\eta,\gamma)$ in $\mathcal{K} = V \times H$. It follows then that defining $\psi$ using the first equation in (3.9), we may, for any $\xi = (\eta,\gamma)$, find an element $\chi = (\varphi,\psi)$ in dom$(A(q))$ which solves (3.9); i.e. for $\lambda > 0$, $R_\lambda(A(q))$ exists as an element in $\mathfrak{L}(\mathcal{K})$.

The coercive inequality (3.12) will be the basis of our convergence arguments.

We consider next a Galerkin type approximation scheme for equation (3.7). As in Section 2, we assume approximation subspaces $H^N \subset V$ satisfying condition (C1). Then for any $w \in V = V \times V$ and each N, there exists $\hat{w}^N \in H^N \times H^N$ satisfying as $N \to \infty$

$$|\hat{w}^N - w|_V \to 0. \tag{3.13}$$

To define the approximating systems, we consider $\sigma(q)$ defined in (3.5) restricted to $\mathcal{H}^N \equiv H^N \times H^N$ and obtain in the usual manner the operators $A^N(q): \mathcal{H}^N \to \mathcal{H}^N$. Since (3.12) holds, we have immediately that $R_\lambda(A^N(q))$ exists in $\mathcal{L}(\mathcal{H}^N)$ for $\lambda > 0$. Denoting by $P^N$ the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}^N$, we may then prove the following convergence results which is analogous to that of Theorem 2.2 for first order systems.

**Theorem 3.1.** *Suppose that conditions* (A), (B) *with* $\lambda_0 = 0$, (C) *for* $\sigma_1$, (A), (B'), (C) *for* $\sigma_2$ *and* (C1) *hold and let* $q^N \to q$ *in* Q. *Then for* $\lambda > 0$ *we have* $R_\lambda(A^N(q^N))P^N\xi \to R_\lambda(A(q))\xi$ *in the V norm for any* $\xi \in \mathcal{H}$.

**Proof.** Let $\xi \in \mathcal{H}$ be arbitrary and put $w = w(q) = R_\lambda(A(q))\xi$, $w^N = w^N(q^N) = R_\lambda(A^N(q^N))P^N\xi$. Let $w = (\varphi,\psi)$, $\xi = (\eta,\gamma)$ so that equation (3.9) is satisfied by this pair. Also, letting $w^N = (\varphi^N,\psi^N)$, $P^N\xi = (\eta^N,\gamma^N)$, we have that this pair satisfies (3.9) with $q = q^N$. In particular, using (3.10) and (3.11) we find for $\zeta \in V$

$$\sigma_\lambda(q)(\varphi,\zeta) = \langle \gamma,\zeta \rangle_H + \lambda \langle \eta,\zeta \rangle_H + \sigma_2(q)(\eta,\zeta)$$
$$\sigma_\lambda(q^N)(\varphi^N,\zeta) = \langle \gamma^N,\zeta \rangle_H + \lambda \langle \eta^N,\zeta \rangle_H + \sigma_2(q^N)(\eta^N,\zeta).$$

Since $w = w(q) \in \text{dom}(A(q)) \subset V$, we may choose $\hat{w}^N = (\hat{\varphi}^N, \hat{\psi}^N)$ in $\mathcal{H}^N$ satisfying (3.13). Thus, choosing $\zeta = \zeta^N \equiv \varphi^N - \hat{\varphi}^N$ in $H^N \subset V$, we may use the above equalities to obtain

$$\sigma_\lambda(q)(\varphi, \zeta^N) - \sigma_\lambda(q^N)(\varphi^N, \zeta^N) = \langle \gamma - \gamma^N, \zeta^N \rangle_H + \lambda \langle n - n^N, \zeta^N \rangle_H \\ + \sigma_2(q)(n, \zeta^N) - \sigma_2(q^N)(n^N, \zeta^N). \tag{3.14}$$

We use this to argue that $\zeta^N \to 0$ in $V$.

Applying the estimate (3.12) to the element $\zeta^N = \varphi^N - \hat{\varphi}^N$ and then using (3.14) we find

$$c_1 |\zeta^N|_V^2 + \hat{\lambda} |\zeta^N|_H^2 \leq \sigma_\lambda(q^N)(\zeta^N, \zeta^N) \tag{3.15}$$
$$= \sigma_\lambda(q^N)(\varphi^N, \zeta^N) - \sigma_\lambda(q)(\varphi, \zeta^N) + \sigma_\lambda(q)(\varphi, \zeta^N) - \sigma_\lambda(q^N)(\hat{\varphi}^N, \zeta^N)$$
$$= \langle \gamma^N - \gamma, \zeta^N \rangle_H + \lambda \langle n^N - n, \zeta^N \rangle_H + \sigma_2(q)(n, \zeta^N) + \sigma_2(q^N)(n^N, \zeta^N)$$
$$+ \sigma_\lambda(q)(\varphi, \zeta^N) - \sigma_\lambda(q^N)(\hat{\varphi}^N, \zeta^N).$$

The last four terms on the right side of this equality may be written as

$$\sigma_2(q)(n^N - n, \zeta^N) - \sigma_2(q)(n^N, \zeta^N) + \sigma_2(q^N)(n^N, \zeta^N) \\ + \sigma_\lambda(q)(\varphi - \hat{\varphi}^N, \zeta^N) + \sigma_\lambda(q)(\hat{\varphi}^N, \zeta^N) - \sigma_\lambda(q^N)(\hat{\varphi}^N, \zeta^N). \tag{3.16}$$

Using the definition (3.11) of $\sigma_\lambda$ and the boundedness and continuity in $q$ of $\sigma_1$, $\sigma_2$ (condition (C) and (A)) along with $|\cdot|_H \leq k |\cdot|_V$ we may bound from above the expression in (3.16) by

$$c_2 |n^N - n|_V |\zeta^N|_V + d(q, q^N) |n^N|_V |\zeta^N|_V + C |\varphi - \hat{\varphi}^N|_V |\zeta^N|_V \\ + (1 + \lambda) d(q, q^N) |\hat{\varphi}^N|_V |\zeta^N|_V. \tag{3.17}$$

where $C \equiv \lambda^2 k^2 + c_2 + \lambda c_2$. We observe that since $P^N$ is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}^N$, hypothesis (C1) implies that $(\eta^N, \gamma^N) = P^N(\eta, \gamma) \to (\eta, \gamma)$ in $\mathcal{H} = V \times H$. Thus, $|\eta^N|_V \leq K$ for some constant K. Since $\hat{\varphi}^N \to \varphi$ in V, we may also assume $|\hat{\varphi}^N|_V \leq K$.

Recalling that $\hat{\lambda} > 0$, we can combine (3.15), (3.16), (3.17) and the above observations to obtain the estimate

$$c_1 |\zeta^N|_V \leq k|\gamma^N - \gamma|_H + (\lambda k + c_2)|\eta^N - \eta|_V + K(2+\lambda)d(q, q^N) + C|\varphi - \hat{\varphi}^N|_V,$$

and thus $\zeta^N \to 0$ in V. That is, $\varphi^N - \hat{\varphi}^N \to 0$ in V and hence $\varphi^N \to \varphi$ in V.

Recalling that $(\varphi^N, \psi^N)$, $(\eta^N, \gamma^N)$ and $(\varphi, \psi)$, $(\eta, \gamma)$ satisfy (3.9) with $q^N$ and q, respectively, we have that

$$\psi^N = \lambda \dot{\varphi}^N - \eta^N \quad \text{and} \quad \psi = \lambda \varphi - \eta.$$

Thus we find immediately that $\psi^N \to \psi$ in V. This completes the proof that $w^N = (\varphi^N, \psi^N) \to w = (\varphi, \psi)$ in the $V$ norm.

Let $T^N(t; q^N)$ denote the $C_0$ semigroup generated by $A^N(q^N)$ in $\mathcal{H}^N$. Then applying Theorem 2.1 with $X = \mathcal{H}$, $X^N = \mathcal{H}^N$ and $A^N = A^N(q^N)$, we obtain immediately from the above discussions the following desired convergence results.

Theorem 3.2. *Under the hypotheses of Theorem 3.1, we have for each $\xi \in \mathcal{H}$ and $t > 0$, $T^N(t; q^N)P^N\xi \to T(t; q)\xi$ in $\mathcal{H}$, with the convergence being uniform in t on compact subintervals.*

The needed convergence results for parameter estimation problems follows

directly from this theorem. In particular, for solutions of (3.7) and their approximations we have (assuming, of course, appropriate smoothness in q of initial data and the perturbation function F)

$$u^N(t;q^N) \to u(t;q) \quad \text{in V norm,}$$
$$v^N(t;q^N) \to v(t;q) \quad \text{in H norm.}$$

To illustrate the types of problems that can be investigated in the context of the theoretical framework developed in this section, we consider a cantilevered Euler-Bernoulli beam of unit length and unit linear mass density. The transverse vibrations can be described by an equation of the form (3.1) with the operators $A(q)$ and $B(q)$ chosen appropriately (see e.g. [BCR], [BR2], [R] for detailed equations). We assume that the end at $x = 0$ is fixed, the end at $x = 1$ is free. Then the state spaces can be chosen as $H = H^0(0,1)$ and $V = H_L^2(0,1) = \{\varphi \in H^2(0,1) \mid \varphi(0) = D\varphi(0) = 0\}$. The stiffness sesquilinear form $\sigma_1$ is given by

$$\sigma_1(q)(\varphi,\psi) = \int_0^1 EID^2\varphi D^2\psi$$

where we assume the stiffness coefficient $q_1(x) = EI(x)$ satisfies $q_1(x) \geq \alpha > 0$ and is in $L_\infty(0,1)$. The operator $A(q)$ in (3.1) is given by $A(q)\varphi = D^2(EID^2\varphi)$ with

$$\text{dom } A(q) = \{\varphi \in V \mid EID^2\varphi \in H^2(0,1), EID^2\varphi(1) = D(EID^2\varphi)(1) = 0\}.$$

If we denote by $q_2$ the damping parameter to be discussed in the following examples, then the admissible parameter set Q can be taken as a compact subset of

$$\tilde{Q} = \{q = (q_1, q_2) \mid q_i \in L_\infty(0,1), q_1(x) \geqslant \alpha > 0\}.$$

With this formulation, a number of important damping mechanisms can be readily treated.

Viscous damping: In this case air or fluid damping is usually assumed proportional to the velocity of displacement so that the term $B(q)\dot{u}(t)$ in (3.1) has the form $b\dot{u}(t)$ for $q_2(x) = b(x)$. The damping sesquilinear form $\sigma_2$ is given by

$$\sigma_2(q)(\varphi,\psi) = \langle b\varphi, \psi \rangle_H$$

which, for $b(x) \geqslant 0$, satisfies the H-semicoercivity condition (B') but of course *is not V-coercive. The domain of the operator in (3.8) is given by* dom $A(q) = $ dom $A(q) \times V$.

Kelvin-Voigt damping: For these models the damping moment is postulated as being proportional to the strain rate and hence the damping term has the form $D^2(c_D ID^2 \dot{u}(t))$ where $q_2(x) = c_D I(x)$. The associated sesquilinear form is given by

$$\sigma_2(q)(\varphi,\psi) = \langle c_D ID^2\varphi, D^2\psi \rangle_H$$

which for $c_D I(x) \geqslant \beta > 0$ satisfies a V-coercive condition and hence of course the H-semicoercivity of hypothesis (B'). Letting $M = EID^2\varphi + c_D ID^2\psi$ denote the "moment" of the beam, we have the domain of $A(q)$ given by

$$\text{dom } A(q) = \{(\varphi,\psi) \in V \times H \mid \psi \in V, M \in H^2(0,1), M(1) = DM(1) = 0\}.$$

If we only have $c_D I(x) \geqslant 0$, then only condition (B') is satisfied but our

theoretical framework is still applicable. We note that results of the framework then provide a distinct improvement over the results for the problems given in [BCR] (where smoothness assumptions on solutions must be hypothesized as well as $q_2 > 0$) and [KG] (where a stronger and less readily characterized topology than that of $C(0,1)$ or $L_\infty(0,1)$ must be used for the compactness assumption on Q).

<u>Spatial hysteresis damping</u>: This damping model, which has been suggested recently and investigated by Russell in [RU], is based on physical foundations that appear particularly appropriate for fiber reinforced composite beams, e.g., beams of composite materials in which strands of fibers are lengthwise bound in an epoxy matrix. The damping term has the form $B(q)\dot{u}(t) = D((G(q) - \nu I)D\dot{u}(t))$ where $G(q)$ is a compact operator in $H^0(0,1)$ defined by a symmetric kernel $q_2(x,y) = b(x,y) = b(y,x) \geqslant 0$ as

$$(G\varphi)(x) = \int_0^1 b(x,y)\varphi(y)dy, \quad \varphi \in H^0(0,1)$$

with $b \in H^0((0,1) \times (0,1))$ and $\nu(x) \equiv \int_0^1 b(x,y)dy$. The associated sesquilinear form is given by

$$\sigma_2(q)(\varphi,\psi) = \langle(\nu I - G)D\varphi, D\psi\rangle_H$$

which is readily seen to be H-semicoercive but not V-coercive. The generator (3.8) for the semigroup has domain

$$\text{dom } A(q) = \{(\varphi,\psi) \in V \times H \mid \psi \in V, \text{ } EID^2\varphi(1) = 0,$$

$$[D(EID^2\varphi) + (\nu I - G)D\psi](1) = 0\}.$$

<u>Time hysteresis damping</u>: These models, referred to as Boltzmann damping models, have been widely studied in recent years (see e.g. [F], [BF], [HW] and the references therein) in connection with flexible structures. In an Euler-Bernoulli beam as formulated above, the damping term $B(q)\dot{u}(t)$ can be replaced by a term that has the form

$$-D^2 \int_{-r}^{0} g(s)D^2u(t+s,\cdot)ds$$

where $q_2 = g(s) = \alpha e^{\beta s}/\sqrt{-s}$, $\alpha,\beta > 0$. This means that equation (3.1) becomes a functional-partial differential equation or partial differential equation with delay or hereditary term. Thus the above theoretical framework is not directly applicable. However, as is shown in [BFW], the framework developed in this section can be appropriately modified and extended to give a succinct theoretical treatment of approximation methods involving estimate convergence and continuous dependence on observations for these models also.

We have used methods based on the theoretical ideas in this section to successfully estimate damping in a number of flexible structure experiments. To date we have studied viscous, Kelvin-Voigt, and Boltzmann damping in vibrations of composite beams [BWIC], [BFW].

Finally, we note that the ideas presented in this paper can be extended to provide a framework for the treatment of nonlinear distributed parameter systems. The coercivity conditions are replaced by monotonicity assumptions and, of course, nonlinear semigroups (evolution systems) and a nonlinear Trotter-Kato approximation theorem play fundamental roles in development of this theory. Details can be found in a forthcoming manuscript [BRR].

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| 16. Abstract |
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| We present a theoretical framework that can be used to treat approximation techniques for very general classes of parameter estimation problems involving distributed systems that are either first or second order in time. Using the approach developed, one can obtain both convergence and stability (continuous dependence of parameter estimates with respect to the observations) under very weak regularity and compactness assumptions on the set of admissible parameters. This unified theory can be used for many problems found in the recent literature and in many cases offers significant improvements to existing results. |

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